GENERIC CHARACTER SHEAVES ON GROUPS OVER $\mathbf{k}[\epsilon]/(\epsilon^r)$

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Introduction

0.1. Let $\mathbf{k}$ be an algebraic closure of the finite field $\mathbf{F}_q$ with q elements where q is a power of a prime number p. Let G be a connected reductive group over $\mathbf{k}$ with a fixed split $\mathbf{F}_q$-rational structure, a fixed Borel subgroup B defined over $\mathbf{F}_q$, with unipotent radical U and a fixed maximal torus T of B defined over $\mathbf{F}_q$. Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{n}$ be the Lie algebras of G, B, T, U. We fix a prime number $l \ne p$. If $\lambda : T(\mathbf{F}_q) \to \bar{\mathbf{Q}}_l^*$ is a character, we can lift λ to a character $\tilde{\lambda} : B(\mathbf{F}_q) \to \bar{\mathbf{Q}}_l^*$ trivial on $U(\mathbf{F}_q)$ and we can form the induced representation $\text{ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)} \tilde{\lambda}$ of $G(\mathbf{F}_q)$. Its character is the class function $G(\mathbf{F}_q) \to \bar{\mathbf{Q}}_l$ given by

(a)
$$y \mapsto \sum_{\substack{B(\mathbf{F}_q)x \in B(\mathbf{F}_q)\backslash G(\mathbf{F}_q);\\ xyx^{-1} \in B(\mathbf{F}_q)}} \tilde{\lambda}(xyx^{-1}).$$

This class function has a geometric analogue. Namely, we consider the diagram $T \xleftarrow{h} \tilde{G} \xrightarrow{\pi} G$ where $\tilde{G} = \{(Bx, y) \in (B\backslash G) \times G; xyx^{-1} \in B\}$, $\pi(Bx, y) = y$ is the Springer map and $h(Bx, y) = d(xyx^{-1})$; here $d : B \to T$ is the obvious homomorphism with kernel U. Let $\mathcal{E}$ be a fixed $\bar{\mathbf{Q}}_l$-local system of rank 1 on T such that $\mathcal{E}^{\otimes m} \cong \bar{\mathbf{Q}}_l$ for some $m \ge 1$ prime to p. The geometric analogue of (a) is the complex $L_1 = \pi_! h^* \mathcal{E} \in \mathcal{D}(G)$. (For any algebraic variety X over $\mathbf{k}$, $\mathcal{D}(X)$ denotes the bounded derived category of constructible $\bar{\mathbf{Q}}_l$-sheaves on X.) When $\mathcal{E}$ is defined over $\mathbf{F}_q$ and has characteristic function λ then L_1 is defined over $\mathbf{F}_q$ and its characteristic function is (up to a nonzero scalar factor) the function (a). Thus L_1 can be viewed as a categoryfied version of the function (a). More precisely, L_1 is (up to shift) a perverse sheaf on G; indeed, one of the main observations of [L1] was that the (proper) map π is small, which implies that L_1 is an intersection cohomology complex; this was the starting point of the theory of character sheaves on G, see [L2]. This point of view is useful since the complexes L_1 are defined independently of the $\mathbf{F}_q$-structure and from them one can extract not only the characters (a) for any q but even their twisted versions defined in [DL].

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0.2. For any integer $r \geq 1$ we consider the ring $\mathbf{k}_r = \mathbf{k}[\epsilon]/(\epsilon^r)$ (ϵ is an indeterminate). Let $G_r = G(\mathbf{k}_r)$ be the group of points of G with values in $\mathbf{k}_r$, viewed as an algebraic group over $\mathbf{k}$ of dimension $r\Delta$ where $\Delta = \dim G$. Let $B_r = B(\mathbf{k}_r)$, $T_r = T(\mathbf{k}_r)$, $U_r = U(\mathbf{k}_r)$. Note that G_r inherits from G a natural $\mathbf{F}_q$-structure and that B_r, T_r, U_r are defined over $\mathbf{F}_q$. For $r = 1$, G_r reduces to G; we would like to extend as much as possible the results in 0.1 from $r = 1$ to a general r. If $\lambda : T_r(\mathbf{F}_q) \to \bar{\mathbf{Q}}_l^*$ is a character, we can lift λ to a character $\tilde{\lambda} : B_r(\mathbf{F}_q) \to \bar{\mathbf{Q}}_l^*$ trivial on $U_r(\mathbf{F}_q)$ and we can form the induced representation $\mathrm{ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)} \tilde{\lambda}$ of $G(\mathbf{F}_q)$. Its character is the class function $G_r(\mathbf{F}_q) \to \bar{\mathbf{Q}}_l$ given by

(a)
$$g' \mapsto \sum_{\substack{B_r(\mathbf{F}_q)g \in B_r(\mathbf{F}_q)\backslash G_r(\mathbf{F}_q); \\ gg'g^{-1} \in B_r(\mathbf{F}_q)}} \tilde{\lambda}(gg'g^{-1}).$$

It generalizes the function 0.1(a). Again this class function has a geometric analogue. Namely, we consider the diagram $T_r \xleftarrow{h_r} \tilde{G}_r \xrightarrow{\pi_r} G_r$ where

$$\tilde{G}_r = \{(B_r g, g') \in (B_r \backslash G_r) \times G_r; gg'g^{-1} \in B_r\},$$

$$\pi_r(B_r g, g') = g', h_r(B_r g, g') = d_r(gg'g^{-1});$$

here $d_r : B_r \to T_r$ is the obvious homomorphism with kernel U_r. We can identify $T = \mathcal{Y} \otimes \mathbf{k}^*$ where $\mathcal{Y}$ is the lattice of one parameter subgroups of T and $T(\mathbf{k}_r) = \mathcal{Y} \otimes \mathbf{k}_r^*$ where $\mathbf{k}_r^*$ is the group of units of $\mathbf{k}_r$. The isomorphism $\mathbf{k}^* \times \mathbf{k}^{r-1} \xrightarrow{\sim} \mathbf{k}_r^*$,

$$(a_0, a_1, \dots, a_{r-1}) \mapsto a_0 + a_1\epsilon + \dots + a_{r-1}\epsilon^{r-1},$$

identifies $T(\mathbf{k}_r)$ with $T \times \mathfrak{t}^{r-1}$. Let $f_1, \dots, f_{r-1}$ be linear functions $\mathfrak{t} \to \mathbf{k}$ and let $\mathcal{E}$ be as in 0.1. We can form the local system $\mathcal{E} \boxtimes \mathcal{L}_{f_1} \boxtimes \dots \boxtimes \mathcal{L}_{f_{r-1}}$ on $T \times \mathfrak{t}^{r-1} = T(\mathbf{k}_r)$ (for the notation $\mathcal{L}_{f_i}$ see 0.3). The geometric analogue of (a) is the complex $L = \pi_{r!} h_r^*(\mathcal{E} \boxtimes \mathcal{L}_{f_1} \boxtimes \dots \boxtimes \mathcal{L}_{f_{r-1}}) \in \mathcal{D}(G_r)$. Again from the complexes L one can extract the characters (a) for any q. In this paper we are interested in the conjecture in [L3, 8(a)] according to which, when $r \geq 2$, L is (up to shift) an intersection cohomology complex on G_r, provided that f_{r-1} is sufficiently general. This would imply that there is a theory of generic character sheaves on G_r. The conjecture was proved in [L3, no.12] in the case where $G = GL_2$ and $r = 2$.

In this paper we give a method to attack the conjecture for any G and even r (but with some restriction on p); we carry out the method in detail in the cases where $r = 2$ and $r = 4$ and we prove the conjecture in these cases (with some restriction on p). We also prove a weak form of the conjecture assuming that $r = 3$ (see Theorem 4.7). I believe that the method of this paper should be applicable with any $r \geq 2$.

Our method is to first replace L by another complex K which is a geometric (categorified) form of the character of a representation constructed by Gérardin

[Ge] in 1975, then to try to describe explicitly the Fourier-Deligne transform of K on G_r (viewed as a vector bundle over G). For $r = 2$ and $r = 4$ we show that this is a simple perverse sheaf of a very special kind, namely one associated to a local system of rank 1 on a closed smooth irreducible subvariety of G_r; by a result of Laumon, this implies that K is itself a simple perverse sheaf, up to twist. Finally, we show that L is a shift of K for the values of r that we consider and this gives the desired result.

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0.3. Notation. In this paper all algebraic varieties are over $\mathbf{k}$. We fix a nontrivial homomorphism $\psi : \mathbf{F}_p \to \bar{\mathbf{Q}}_l^*$. For any morphism $f : X \to \mathbf{k}$ let $X_f = \{(x, \lambda) \in X\times\mathbf{k}; \lambda^q - \lambda = f(x)\}$ and let $\iota : X_f \to X$ be the Artin-Schreier covering $(x, \lambda) \mapsto x$. Then $\iota_!\bar{\mathbf{Q}}_l$ is a local system with a natural action of $\mathbf{F}_p$ (coming from the $\mathbf{F}_p$-action $\zeta : (x, \lambda) \mapsto (x, \lambda + \zeta)$ on $\tilde{X}$); we denote by $\mathcal{L}_f$ the ψ-eigenspace of this action (a local system of rank 1 on X).

Let $\delta = \dim T$.

For $x \in G$ if X is an element of $\mathfrak{g}$ or a subset of $\mathfrak{g}$ we write ${}^xX$ instead of $\mathrm{Ad}(x)X$ and ${}_xX$ instead of $\mathrm{Ad}(x^{-1})X$.

1. The complex K

1.1. Let $X_1, X_2, \dots$ and $Y_1, Y_2, \dots$ be two sequences of noncommuting indeterminates. From the Campbell-Baker-Hausdorff formula we deduce the equality

$$(e^{\epsilon X_1}e^{\epsilon^2 X_2}\dots)(e^{\epsilon Y_1}e^{\epsilon^2 Y_2}\dots) = e^{\epsilon z_1}e^{\epsilon^2 z_2}\dots$$

where $z_i = z_i(X_1, X_2, \dots, X_i, Y_1, Y_2, \dots, Y_i)$, $(i \ge 1)$ are universal Lie polynomials with coefficients in $\mathbf{Z}[(i!)^{-1}]$. (Here ϵ commutes with each X_i, Y_i.) For example,

$z_1(X_1, Y_1) = X_1 + Y_1$,

$z_2(X_1, X_2, Y_1, Y_2) = X_2 + Y_2 + [X_1, Y_1]/2$,

$z_3(X_1, X_2, X_3, Y_1, Y_2, Y_3) = X_3+Y_3+[X_2, Y_1]-[X_1, [X_1, Y_1]]/6-[Y_1, [X_1, Y_1]]/3$.

We deduce that if $X_1, X_2, \dots,$ $X'_1, X'_2, \dots$ and $Y_1, Y_2, \dots$ are three sequences of noncommuting indeterminates then we have the equality

$$(e^{\epsilon X'_1}e^{\epsilon^2 X'_2}\dots)(e^{\epsilon Y_1}e^{\epsilon^2 Y_2}\dots)(e^{\epsilon X_1}e^{\epsilon^2 X_2}\dots)^{-1} = e^{\epsilon u_1}e^{\epsilon^2 u_2}\dots$$

where $u_i = u_i(X'_1, \dots, X'_i, Y_1, \dots, Y_i, X_1, \dots, X_i)$, $(i \ge 1)$ are universal Lie polynomials with coefficients in $\mathbf{Z}[(i!)^{-1}]$. For example,

$$u_1(X_1, Y_1, X'_1) = X'_1 - X_1 + Y_1,$$

$u_2(X_1, X_2, Y_1, Y_2, X'_1, X'_2) = X'_2 - X_2 + Y_2 + [X'_1, Y_1]/2 - [X'_1, X_1]/2 - [Y_1, X_1]/2,$

$u_3(X_1, X_2, X_3, Y_1, Y_2, Y_3, X'_1, X'_2, X'_3)$
$= X'_3 - X_3 + Y_3 + [X'_2, Y_1] + [X_2, X_1] - [X'_2, X_1] - [Y_2, X_1] - [X'_1, [X'_1, Y_1]]/6$
$- [Y_1, [X'_1, Y_1]]/3 + [X_1, [X'_1, Y_1]]/2 + [X'_1, [X'_1, X_1]]/6 + [X'_1, [Y_1, X_1]]/6$
$+ [Y_1, [X'_1, X_1]]/6 + [Y_1, [Y_1, X_1]]/6 - [X_1, [X'_1, X_1]]/3 - [X_1, [Y_1, X_1]]/3.$

Note that
(a) $u_i(X'_1, \dots, X'_i, Y_1, \dots, Y_i, X_1, \dots, X_i) = X'_i - X_i + Y_i + u'_i$ where
$u'_i = u'_i(X'_1, \dots, X'_{i-1}, Y_1, \dots, Y_{i-1}, X_1, \dots, X_{i-1})$
is a Lie polynomial in $X'_1, \dots, X'_{i-1}, Y_1, \dots, Y_{i-1}, X_1, \dots, X_{i-1}$.

1.2. We now fix $r \ge 2$. We write $r = 2r'$ if r is even and $r = 2r'+1$ if r is odd. We always assume that $p \ge r$. Then for any $X \in \mathfrak{g}$ and any $m \ge 1$, the exponential $e^{\epsilon^m X} \in G_r$ is well defined. For any $X_1, X_2, \dots, X_{r-1}$ in $\mathfrak{g}$ we set

$$|X_1, X_2, \dots, X_{r-1}| = e^{\epsilon X_1} e^{\epsilon^2 X_2} \dots e^{\epsilon^{r-1} X_{r-1}} \in G_r.$$

We have an isomorphism of algebraic varieties

$$G \times \mathfrak{g}^{r-1} \xrightarrow{\sim} G_r$$

given by

$$(x, X_1, X_2, \dots, X_{r-1}) \mapsto x|X_1, X_2, \dots, X_{r-1}| = |{}_xX_1, {}_xX_2, \dots, {}_xX_{r-1}|x.$$

This restricts to isomorphisms of algebraic varieties $B \times \mathfrak{b}^{r-1} \xrightarrow{\sim} B_r$, $U \times \mathfrak{n}^{r-1} \xrightarrow{\sim} U_r$, $T \times \mathfrak{t}^{r-1} \xrightarrow{\sim} T_r$. (The last isomorphism is the same as one in 0.2.)

Let $X_1, X_2, \dots, X_{r-1}$ and $Y_1, Y_2, \dots, Y_{r-1}$ be two sequences in $\mathfrak{g}$ and let x, y be in G. We have

$$(x|X_1, \dots, X_{r-1}|)(y|Y_1, \dots, Y_{r-1}|) = xy|Z_1, \dots, Z_{r-1}|$$

where $Z_i = z_i({}_yX_1, \dots, {}_yX_i, Y_1, \dots, Y_i) \in \mathfrak{g}$ $(i = 1, \dots, r-1)$ with notation of 1.1 and where $[,]$ becomes the Lie bracket in $\mathfrak{g}$; note that Z_i are well defined since $p \ge r$. Moreover, we have

$$(x|X_1, \dots, X_{r-1}|)(y|Y_1, \dots, Y_{r-1}|)(x|X_1, \dots, X_{r-1}|)^{-1} = xyx^{-1}|U_1, \dots, U_{r-1}|$$

where $U_i = {}^xu_i({}_yX_1, \dots, {}_yX_i, Y_1, \dots, Y_i, X_1, \dots, X_i) \in \mathfrak{g}$ $(i = 1, \dots, r-1)$ with notation of 1.1); note that U_i are well defined since $p \ge r$.

1.3. Let $\phi : E \to X$ be an algebraic vector bundle with fibres of constant dimension N. Let $f : E \to \mathbf{k}$ be a morphism such that for any $x \in X$ the restriction $f^x : \phi^{-1}(x) \to \mathbf{k}$ is affine linear. Let X_0 be the set of all $x \in X$ such that f^x is a

constant (depending of x) and let $f_0 : X_0 \to \mathbf{k}$ be such that $f(e) = f_0(\phi(e))$ for all $e \in \phi^{-1}(X_0)$. Let $j : X_0 \to X$ be the (closed) imbedding. We show:

(a) $\phi_!\mathcal{L}_f \cong j_!\mathcal{L}_{f_0}[-2N]$.

For any $x \in X - X_0$ we have $H^i_c(\phi^{-1}(x), \mathcal{L}_f) = 0$ for all i. Hence $\phi_!\mathcal{L}_f|_{X-X_0} = 0$. We are reduced to the case where $X = X_0$. In this case we have $\mathcal{L}_f = \phi^*\mathcal{L}_{f_0}$ hence

$$\phi_!\mathcal{L}_f = \phi_!\phi^*\mathcal{L}_{f_0} = \mathcal{L}_{f_0} \otimes \phi_!\phi^*\bar{\mathbf{Q}}_l \cong \mathcal{L}_{f_0}[-2N],$$

as required. (We ignore Tate twists.)

If in addition we are given a local system $\mathcal{F}$ on X and we denote $\phi^*\mathcal{F}$ and $j^*\mathcal{F}$ again by $\mathcal{F}$, then from (a) we have immediately

(b) $\phi_!(\mathcal{F} \otimes \mathcal{L}_f) \cong j_!(\mathcal{F} \otimes \mathcal{L}_{f_0})[-2N]$.

1.4. In the rest of this paper we assume that a nondegenerate symmetric bilinear invariant form $\langle,\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbf{k}$ is given and that a sequence $A_1, A_2, \dots, A_{r-1}$ of elements of $\mathfrak{t}$ is given such that A_{r-1} is regular semisimple. This requires a further restriction on p in addition to the restriction $p \geq r$.

For a subspace E of $\mathfrak{g}$ we set $E^\perp = \{\xi \in \mathfrak{g}; \langle \xi, E\rangle = 0\}$.

Let $\mathcal{X}$ be the variety of all

$$(Tx, y, X_1, X_2, \dots, X_{r-1}, Y_1, Y_2, \dots, Y_{r-1}) \in (T\backslash G) \times G \times \mathfrak{g}^{2r-2}$$

such that $xyx^{-1} \in T$ and

$$u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j) \in {}_x\mathfrak{t} \text{ for } 1 \leq j \leq r' - 1,$$

$$u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j) \in {}_x\mathfrak{b} \text{ if } j = r' \text{ and } r \text{ is odd.}$$

We have a diagram

$$G_r \xleftarrow{\pi} \mathcal{X} \xrightarrow{h} \mathbf{k}$$

where $\pi(Tx, y, X_1, \dots, X_{r-1}, Y_1, \dots, Y_{r-1}) = y|Y_1, \dots, Y_{r-1}|$,

$$\begin{aligned} &h(Tx, y, X_1, \dots, X_{r-1}, Y_1, \dots, Y_{r-1}) \\ &= \sum_{j \in [1, r-1]} \langle {}_xA_j, u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j)\rangle. \end{aligned}$$

(Note that if x is replaced by tx, $(t \in T)$ in the last sum, the sum remains unchanged since ${}_tA_j = A_j$ for all j.) We define $\iota : \mathcal{X} \to T$ by

$$\iota(Tx, y, X_1, X_2, \dots, X_{r-1}, Y_1, Y_2, \dots, Y_{r-1}) = xyx^{-1}$$

and we set $\tilde{\mathcal{E}} = \iota^*\mathcal{E}$. Let $K = \pi_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_h) \in \mathcal{D}(G_r)$. Via the identification $G_r = G \times \mathfrak{g}^{r-1}$ (see 1.2) we can regard G_r as a vector bundle over G with fibre

$\mathfrak{g}^{r-1}$ endowed with a nondegenerate symmetric bilinear form. Hence the Fourier-Deligne transform $\hat{K} \in \mathcal{D}(G_r)$ along these fibres is well defined. More explicitly, for $i=1,2$ we have the diagram $G_r \xleftarrow{\rho_i} G_r \times_G G_r \xrightarrow{h'} \mathbf{k}$ where ρ_i is the projection to the i-th factor and

$$h'(x|Y_1,\dots,Y_{r-1}|,x|R_1,R_2,\dots,R_{r-1}|) = \sum_{j\in[1,r-1]} \langle Y_j, R_j\rangle.$$

Then $\hat{K} = \rho_{2!}(\rho_1^* K \otimes \mathcal{L}_{h'})[(r-1)\Delta]$ that is,

$$\hat{K} = \rho_{2!}(\rho_1^* \pi_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_h) \otimes \mathcal{L}_{h'})[(r-1)\Delta].$$

Let $\tilde{\mathcal{X}}$ be the variety of all

$$(Tx, y, X_1, X_2, \dots, X_{r-1}, Y_1, Y_2, \dots, Y_{r-1}, R_1, R_2, \dots, R_{r-1}) \in (T\backslash G) \times G \times \mathfrak{g}^{3r-3}$$

such that $xyx^{-1} \in T$ and

$$u_j({}_yX_1,\dots,{}_yX_j, Y_1,\dots,Y_j, X_1,\dots,X_j) \in {}_x\mathfrak{t} \text{ for } 1 \le j \le r'-1,$$

$$u_j({}_yX_1,\dots,{}_yX_j, Y_1,\dots,Y_j, X_1,\dots,X_j) \in {}_x\mathfrak{b} \text{ if } j=r' \text{ and } r \text{ is odd }.$$

We have a cartesian diagram

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \xrightarrow{\tilde{\rho}_1} & \mathcal{X} \\ \sigma\downarrow & & \pi\downarrow \\ G_r \times_G G_r & \xrightarrow{\rho_1} & G_r \end{array}$$

where

$$\tilde{\rho}_1(Tx, y, X_1,\dots,X_{r-1}, Y_1,\dots,Y_{r-1}, R_1,\dots,R_{r-1})$$
$$= (Tx, y, X_1,\dots,X_{r-1}, Y_1,\dots,Y_{r-1}),$$

$$\sigma(Tx, y, X_1,\dots,X_{r-1}, Y_1,\dots,Y_{r-1}, R_1,\dots,R_{r-1})$$
$$= (y|Y_1,\dots,Y_{r-1}|, y|R_1,\dots,R_{r-1}|).$$

It follows that

$$\hat{K} = \rho_{2!}(\sigma_!(\tilde{\mathcal{E}} \otimes \tilde{\rho}_1^* \mathcal{L}_h \otimes \mathcal{L}_{h'})[(r-1)\Delta]$$
$$= \rho_{2!}\sigma_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{\tilde{h}'} \otimes \mathcal{L}_{\tilde{h}''}) = (\rho_2\sigma)_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{\tilde{h}})[(r-1)\Delta]$$

where $\tilde h''=h\mathrm{tr}_1:\tilde{\mathcal{X}}\to\mathbf{k}$, $\tilde h'=h'\sigma:\tilde{\mathcal{X}}\to\mathbf{k}$, $\tilde h=\tilde h'+\tilde h'':\tilde{\mathcal{X}}\to\mathbf{k}$ and the inverse image of $\tilde{\mathcal{E}}$ under $\tilde{\mathcal{X}}\to T$,

$$(Tx,y,X_1,\dots,X_{r-1},Y_1,\dots,Y_{r-1},R_1,\dots,R_{r-1})\mapsto xyx^{-1}$$

is denoted again by $\tilde{\mathcal{E}}$. Thus,

$$\hat K=\pi'_!(\tilde{\mathcal{E}}\otimes\mathcal{L}_{\tilde h})[(r-1)\Delta]$$

where $\pi':\tilde{\mathcal{X}}\to G_r$ and $\tilde h:\tilde{\mathcal{X}}\to\mathbf{k}$ are given by

$$\pi'(Tx,y,X_1,\dots,X_{r-1},Y_1,\dots,Y_{r-1},R_1,\dots,R_{r-1})=y|R_1,\dots,R_{r-1}|,$$

$$\begin{aligned}&\tilde h(Tx,y,X_1,\dots,X_{r-1},Y_1,\dots,Y_{r-1},R_1,\dots,R_{r-1})\\&=\sum_{j\in[1,r-1]}\langle Y_j,R_j\rangle+\sum_{j\in[1,r-1]}\langle {}_xA_j,u_j({}_yX_1,\dots,{}_yX_j,Y_1,\dots,Y_j,X_1,\dots,X_j)\rangle.\end{aligned}$$

1.5. Let $\tilde{\mathcal{X}}''$ be the variety of all

$$\begin{aligned}&(Tx,y,X_1,X_2,\dots,X_{r-2},Y_1,Y_2,\dots,Y_{r-2},R_1,R_2,\dots,R_{r-1})\\&\in(T\backslash G)\times G\times\mathfrak{g}^{(r-2)+(r-2)+(r-1)}\end{aligned}$$

such that $xyx^{-1}\in T$ and

$$u_j({}_yX_1,\dots,{}_yX_j,Y_1,\dots,Y_j,X_1,\dots,X_j)\in{}_x\mathfrak{t}\text{ for }1\le j\le r'-1,$$

$$u_j({}_yX_1,\dots,{}_yX_j,Y_1,\dots,Y_j,X_1,\dots,X_j)\in{}_x\mathfrak{b}\text{ if }j=r'\text{ and }r\text{ is odd .}$$

(The equations make sense since if $1\le j\le r'-1$ then $j\le r-2$ and since when r is odd we have $r'=r-r'-1\le r-2$.) We define $\mu:\tilde{\mathcal{X}}\to\tilde{\mathcal{X}}''$ by

$$\begin{aligned}&(Tx,y,X_1,X_2,\dots,X_{r-1},Y_1,Y_2,\dots,Y_{r-1},R_1,R_2,\dots,R_{r-1})\mapsto\\&(Tx,y,X_1,X_2,\dots,X_{r-2},Y_1,Y_2,\dots,Y_{r-2},R_1,R_2,\dots,R_{r-1}).\end{aligned}$$

This is a vector bundle; for a fixed

$$s=(Tx,y,X_1,X_2,\dots,X_{r-2},Y_1,Y_2,\dots,Y_{r-2},R_1,R_2,\dots,R_{r-1})\in\tilde{\mathcal{X}}'',$$

the fibre $\mu^{-1}(s)$ can be identified with $\mathfrak{g}^2$ with coordinates X_{r-1},Y_{r-1}. The restriction of $\tilde h$ to $\mu^{-1}(s)$ is of the form

$$(X_{r-1},Y_{r-1})\mapsto\langle Y_{r-1},R_{r-1}+{}_xA_{r-1}+c$$

where c is a constant depending on s. We use that $\langle {}_xA_j, {}_yX_{r-1} - X_{r-1}\rangle = 0$; this holds since ${}^{yx^{-1}}A_{r-1} = {}^{x^{-1}}A_{r-1}$ (recall that $xyx^{-1} \in T$). Thus this restriction is affine linear and is constant precisely when $R_{r-1} = -{}_xA_{r-1}$. Hence the results in 1.3 are applicable. Let $\bar{\mathcal{X}}$ be the variety of all

$$(Tx, y, X_1, X_2, \dots, X_{r-2}, Y_1, Y_2, \dots, Y_{r-2}, R_1, R_2, \dots, R_{r-1}) \in \tilde{\mathcal{X}}''$$

such that $R_{r-1} = -{}_xA_{r-1}$. We define $\bar{\pi} : \bar{\mathcal{X}} \to G_r$, $\bar{h} : \bar{\mathcal{X}} \to \mathbf{k}$ by

$$\begin{aligned}&\bar{\pi}(Tx, y, X_1, X_2, \dots, X_{r-2}, Y_1, Y_2, \dots, Y_{r-2}, R_1, R_2, \dots, R_{r-1})\\ &= y|R_1, R_2, \dots, R_{r-1}|,\end{aligned}$$

$$\begin{aligned}&\bar{h}(Tx, y, X_1, X_2, \dots, X_{r-2}, Y_1, Y_2, \dots, Y_{r-2}, R_1, R_2, \dots, R_{r-1})\\ &= \sum_{j\in[1,r-2]} \langle Y_j, R_j\rangle + \sum_{j\in[1,r-2]} \langle {}_xA_j, u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j)\rangle\\ &+ \langle {}_xA_{r-1}, u'_{r-1}({}_yX_1, \dots, {}_yX_{r-2}, Y_1, \dots, Y_{r-2}, X_1, \dots, X_{r-2})\rangle,\end{aligned}$$

with notation of 1.1(a). The inverse image of $\mathcal{E}$ under $\bar{\mathcal{X}} \to T$,

$$(Tx, y, X_1, X_2, \dots, X_{r-2}, Y_1, Y_2, \dots, Y_{r-2}, R_1, R_2, \dots, R_{r-1}) \mapsto xyx^{-1}$$

is denoted again by $\tilde{\mathcal{E}}$. Then from 1.3(b) we deduce

(a) $$\hat{K} = \bar{\pi}_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{\bar{h}})[(r-5)\Delta].$$

Let $\mathcal{C} \subset \mathfrak{g}$ be the G-orbit of $-A_{r-1}$ for the adjoint action, a regular semisimple orbit. Let $V = \{(y, R) \in G \times \mathcal{C}; {}^yR = R\}$. Let Σ be the support of $\hat{K}$ (a closed subset of G_r). From (a) we see that

$$\Sigma \subset \{y|R_1, R_2, \dots, R_{r-1}| \in G_r; (y, R_{r-1}) \in V\}.$$

It is likely that Σ is a smooth subvariety of G_r, isomorphic to a vector bundle over V with fibres isomorphic to $(\mathfrak{t}^\perp)^{r-2}$. We will we show that this is the case at least when $r \in \{2, 3, 4\}$. Moreover, it is likely that when r is even, $\hat{K}$ is up to shift the intersection cohomology complex associated to a local system of rank 1 on the smooth closed subvariety Σ. We will show that this is the case when $r \in \{2, 4\}$ and that the analogous statement is not true when $r = 3$.

1.6. The method used in 1.5 to eliminate the variables X_{r-1}, Y_{r-1} can be used to eliminate all variables $X_{r-r'}, \dots, X_{r-1}, Y_{r-r'}, \dots, Y_{r-1}$. Let $\tilde{\mathcal{X}}''_1$ be the variety of all

$$\begin{aligned}&(Tx, y, X_1, X_2, \dots, X_{r-r'-1}, Y_1, Y_2, \dots, Y_{r-r'-1}, R_1, R_2, \dots, R_{r-1})\\ &\in (T\backslash G) \times G \times \mathfrak{g}^{r-1+2(r-r'-1)}\end{aligned}$$

such that $xyx^{-1} \in T$ and

$$u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j) \in {}_x\mathfrak{t} \text{ for } 1 \le j \le r' - 1,$$
$$u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j) \in {}_x\mathfrak{b} \text{ if } j = r' \text{ and } r \text{ is odd }.$$

(The equations make sense since if $1 \le j \le r' - 1$ then $j \le r - r' - 1$ and since when r is odd we have $r' = r - r' - 1$.) We define $\mu_1 : \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}_1''$ by

$$(Tx, y, X_1, X_2, \dots, X_{r-1}, Y_1, Y_2, \dots, Y_{r-1}, R_1, R_2, \dots, R_{r-1}) \mapsto$$
$$(Tx, y, X_1, X_2, \dots, X_{r-r'-1}, Y_1, Y_2, \dots, Y_{r-r'-1}, R_1, R_2, \dots, R_{r-1}).$$

This is a vector bundle; for a fixed

$$s = (Tx, y, X_1, X_2, \dots, X_{r-r'-1}, Y_1, Y_2, \dots, Y_{r-r'-1}, R_1, R_2, \dots, R_{r-1}) \in \bar{\mathcal{X}},$$

the fibre $\mu^{-1}(s)$ can be identified with $\mathfrak{g}^{2r'}$ with coordinates

$$X_{r-r'}, do, X_{r-1}, Y_{r-r'}, \dots, Y_{r-1}.$$

The restriction of $\tilde{h}$ to $\mu^{-1}(s)$ in an affine linear function. This follows from the fact that for $j \in [1, r-1]$, the Lie polynomial

$$u_j(X'_1, \dots, X'_j, Y_1, \dots, Y_j, X_1, \dots, X_j)$$

is a linear combination of terms which are iterated brackets of indeterminates X'_h, Y_h, X_h with sum of indices equal to j (hence $\le r-1$) hence containing at most one X'_h, Y_h or X_h with $h \ge r - r'$. (If they contained more than one, we would have $2(r-r') \le r-1$ hence $r \le 2r'-1$, a contradiction.) Hence the results in 1.3 are applicable and they result in a description of $\hat{K}$ which does not involve $X_{r-r'}, \dots, X_{r-1}, Y_{r-r'}, \dots, Y_{r-1}$. But even after this method is applied, one needs further arguments to analyze $\hat{K}$, as we will see in Sections 2 and 3.

2. The cases $r = 2$ and $r = 4$

2.1. In this subsection we assume that $r = 2$. Now

$$\bar{\mathcal{X}} = \{(Tx, y, R_1) \in (T\backslash G) \times G \times \mathfrak{g}; xyx^{-1} \in T, R_1 = -{}_xA_1.\}$$

We have $\bar{\pi}(Tx, y, R_1) = y|R_1|$ and $\bar{h} : \bar{\mathcal{X}} \to \mathbf{k}$ is identically 0. Using 1.5(a) we have

(a) $$\hat{K} = \bar{\pi}_! \tilde{\mathcal{E}}[-3\Delta]$$

Note that $\bar{\pi}$ defines an isomorphism of $\bar{\mathcal{X}}$ with

$$\mathcal{Z} = \{y|R_1| \in G_2; R_1 \in \mathcal{C}, {}^yR_1 = R_1\}$$

and that $\mathcal{Z}$ is closed in G_2 (we use that $\mathcal{C}$ is closed in $\mathfrak{g}$). Moreover $\mathcal{Z}$ is a smooth subvariety of G_2 and $\bar{\pi}$ can be viewed as the imbedding $\mathcal{Z} \to G_2$. Since $\mathcal{Z}$ is closed in G_2 and smooth, irreducible of dimension Δ we see that $\bar{\pi}_! \tilde{\mathcal{E}}[\Delta]$ is a simple perverse sheaf on G_2. Hence $\hat{K}[4\Delta]$ is a simple perverse sheaf on G_2 with support $\Sigma = \mathcal{Z}$. Using Laumon's theorem [La], it follows that

(b) $K[4\Delta]$ *is a simple perverse sheaf on* G_2.

2.2. We now assume (until the end of 2.5) that $r=4$. Now $\bar{\mathcal{X}}$ is the variety of all

$$(Tx, y, X_1, X_2, Y_1, Y_2, R_1, R_2, R_3) \in (T\backslash G) \times G \times \mathfrak{g}^7$$

such that $xyx^{-1} \in T$, ${}_yX_1 - X_1 + Y_1 \in {}_x\mathfrak{t}$ and $R_3 = -{}_xA_3$. We have

$$\bar{\pi}(Tx, y, X_1, X_2, Y_1, Y_2, R_1, R_2, R_3) = y|R_1, R_2, R_3|,$$

$$\begin{aligned}
&\bar{h}(Tx, y, X_1, X_2, Y_1, Y_2, R_1, R_2, R_3)\\
&= \langle Y_1, R_1\rangle + \langle Y_2, R_2\rangle + \langle {}_xA_1, {}_yX_1 - X_1 + Y_1\rangle\\
&+ \langle {}_xA_2, {}_yX_2 - X_2 + Y_2 + [{}_yX_1, Y_1]/2 - [{}_yX_1, X_1]/2 - [Y_1, X_1]/2\rangle\\
&+ \langle {}_xA_3, [{}_yX_2, Y_1] + [X_2, X_1] - [{}_yX_2, X_1] - [Y_2, X_1] - [{}_yX_1, [{}_yX_1, Y_1]]/6\\
&- [Y_1, [{}_yX_1, Y_1]]/3 + [X_1, [{}_yX_1, Y_1]]/2 + [{}_yX_1, [{}_yX_1, X_1]]/6 + [{}_yX_1, [Y_1, X_1]]/6\\
&+ [Y_1, [{}_yX_1, X_1]]/6 + [Y_1, [Y_1, X_1]]/6 - [X_1, [{}_yX_1, X_1]]/3 - [X_1, [Y_1, X_1]]/3\rangle.
\end{aligned}$$

We make a change of variable $Y_1 = X_1 - {}_yX_1 + {}_x\tau$ where $\tau \in \mathfrak{t}$. Then $\bar{\mathcal{X}}$ becomes the variety of all

$$(Tx, y, \tau, X_1, X_2, Y_2, R_1, R_2, R_3) \in (T\backslash G) \times G \times \mathfrak{t} \times \mathfrak{g}^6$$

such that $xyx^{-1} \in T$ and and $R_3 = -{}_xA_3$. Now $\bar{\pi} : \bar{\mathcal{X}} \to G_4$ and $\bar{h} : \bar{\mathcal{X}} \to \mathbf{k}$ become

$$\bar{\pi}(Tx, y, \tau, X_1, X_2, Y_2, R_1, R_2, R_3) = y|R_1, R_2, R_3|,$$

$$\begin{aligned}
&\bar{h}(Tx, y, \tau, X_1, X_2, Y_2, R_1, R_2, R_3)\\
&= \langle X_1 - {}_yX_1 + {}_x\tau, R_1\rangle + \langle Y_2, R_2\rangle + \langle {}_xA_1, {}_x\tau\rangle\\
&+ \langle {}_xA_2, {}_yX_2 - X_2 + Y_2 + [{}_yX_1, X_1 + {}_x\tau]/2 - [{}_yX_1, X_1]/2 - [-{}_yX_1 + {}_x\tau, X_1]/2\rangle\\
&+ \langle {}_xA_3, [{}_yX_2, X_1 - {}_yX_1 + {}_x\tau] + [X_2, X_1] - [{}_yX_2, X_1] - [Y_2, X_1]\\
&- [{}_yX_1, [{}_yX_1, X_1 + {}_x\tau]]/6 - [X_1 - {}_yX_1 + {}_x\tau, [{}_yX_1, X_1 + {}_x\tau]]/3\\
&+ [X_1, [{}_yX_1, X_1 + {}_x\tau]]/2 + [{}_yX_1, [{}_yX_1, X_1]]/6 + [{}_yX_1, [-{}_yX_1 + {}_x\tau, X_1]]/6\\
&+ [X_1 - {}_yX_1 + {}_x\tau, [{}_yX_1, X_1]]/6 + [X_1 - {}_yX_1 + {}_x\tau, [-{}_yX_1 + {}_x\tau, X_1]]/6\\
&- [X_1, [{}_yX_1, X_1]]/3 - [X_1, [-{}_yX_1 + {}_x\tau, X_1]]/3\rangle.
\end{aligned}$$

For $i = 1, 2, 3$ we have $[A_i, \tau] = 0$ since $\mathfrak{t}$ is abelian; it follows that $\langle A_i, [\xi, \tau]\rangle = 0$ for any $\xi \in \mathfrak{g}$. We also have $\langle {}_xA_i, {}_yX_j - X_j\rangle = 0$; indeed the left hand side is $\langle {}^{yx^{-1}}A_i - {}^{x^{-1}}A_i, X_j\rangle$ and this is zero since ${}^{xyx^{-1}}A_i = A_i$ (recall that $xyx^{-1} \in T$). Similarly we have $\langle {}_xA_3, [{}_yX_2, -{}_yX_1] + [X_2, X_1]\rangle = 0$; indeed, the left hand side is $\langle {}^{yx^{-1}}A_3 - {}^{x^{-1}}A_3, [X_1, X_2]\rangle = 0$. We see that

$$\begin{aligned}
&\bar{h}(Tx, y, \tau, X_1, X_2, Y_2, R_1, R_2, R_3) = \langle X_1 - {}_yX_1 + {}_x\tau, R_1\rangle\\
&+ \langle Y_2, R_2\rangle + \langle {}_xA_1, {}_x\tau\rangle + \langle {}_xA_2, Y_2 - [-{}_yX_1, X_1]/2\rangle\\
&+ \langle {}_xA_3, -[Y_2, X_1] + [{}_yX_1, [{}_yX_1, {}_x\tau]]/6 + [X_1, [X_1, {}_x\tau]]/6\\
&+ [X_1, [{}_yX_1, {}_x\tau]]/6 + [{}_yX_1, [{}_yX_1, X_1]]/6 + [X_1, [{}_yX_1, X_1]]/6\rangle.
\end{aligned}$$

Next we use the identity

$$\langle {}_xA_3, [Z, [Z', {}_x\tau]]\rangle = \langle {}_x\tau, [Z, [Z', {}_xA_3]]\rangle$$

for any Z, Z' in $\mathfrak{g}$. (This follows from $[{}_xA_3, {}_x\tau] = 0$.) We also use the equality

$$\langle {}_xA_3, [{}_yX_1, [{}_yX_1, {}_x\tau]]\rangle = \langle {}_xA_3, [X_1, [X_1, {}_x\tau]]\rangle.$$

(Since ${}^{yx^{-1}}\tau = {}^{x^{-1}}\tau$, ${}^{yx^{-1}}A_3 = {}^{x^{-1}}A_3$, the left hand side is

$$\langle {}_xA_3, {}_y[X_1, [X_1, {}_x\tau]]\rangle = \langle {}^{yx^{-1}}A_3, [X_1, [X_1, {}_x\tau]]\rangle = \langle {}_xA_3, [X_1, [X_1, {}_x\tau]]\rangle,$$

as required.) We see that

$$\begin{aligned}&\bar{h}(Tx, y, \tau, X_1, X_2, Y_2, R_1, R_2, R_3) = \langle Y_2, R_2 + {}_xA_2 - [X_1, {}_xA_3]\rangle \\ &+ \langle {}_x\tau, R_1 + {}_xA_1 + [X_1, [X_1, {}_xA_3]]/6 + [X_1, [{}_yX_1, {}_xA_3]]/3\rangle + \langle X_1 - {}_yX_1, R_1\rangle \\ &+ \langle {}_xA_2, [{}_yX_1, X_1]/2\rangle + \langle {}_xA_3, [{}_yX_1, [{}_yX_1, X_1]]/6 + [X_1, [{}_yX_1, X_1]]/6\rangle.\end{aligned}$$

2.3. Let $\mathcal{T} = \{(Tx, y, X_1, R_1, R_2, R_3) \in (T\backslash G)\times G\times\mathfrak{g}^4; xyx^{-1} \in T, R_3 + {}_xA_3 = 0\}$. Let $\mathcal{T}_0$ be the closed subset of $\mathcal{T}$ consisting of all $(Tx, y, X_1, R_1, R_2, R_3)$ such that

$$R_2 + {}_xA_2 - [X_1, {}_xA_3] = 0,$$

$$R_1 + {}_xA_1 + [X_1, [X_1, {}_xA_3]]/3 + [X_1, [{}_yX_1, {}_xA_3]]/6 \in ({}_x\mathfrak{t})^\perp.$$

Define $\tilde{h}_0 : \mathcal{T}_0 \to \mathbf{k}$ by

$$\begin{aligned}&\tilde{h}_0(Tx, y, X_1, R_1, R_2, R_3) = \langle X_1 - {}_yX_1, R_1\rangle + \langle {}_xA_2, [{}_yX_1, X_1]/2\rangle \\ &+ \langle {}_xA_3, [{}_yX_1, [{}_yX_1, X_1]]/6 + [X_1, [{}_yX_1, X_1]]/6\rangle.\end{aligned}$$

Define $\bar{\mathcal{X}} \xrightarrow{\phi} \mathcal{T} \xrightarrow{\phi'} G_4$ by

$$\phi(Tx, y, \tau, X_1, X_2, Y_2, R_1, R_2, R_3) = (Tx, y, X_1, R_1, R_2, R_3),$$

$$\phi'(Tx, y, X_1, R_1, R_2, R_3) = y|R_1, R_2, R_3|$$

so that $\pi' = \phi'\phi$. Now ϕ is a vector bundle with fibres of dimension $N = 2\Delta + \delta$. Note that the restriction of $\tilde{h} : \bar{\mathcal{X}} \to \mathbf{k}$ to any fibre of ϕ is affine linear and this restriction is constant precisely at the fibres over points in $\mathcal{T}_0$; moreover the constant is given by the value of $\tilde{h}_0$. Using 1.3(b), we see that

(a) $$\hat{K} = j_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{\tilde{h}_0})[-5\Delta - 2\delta]$$

where $j : \mathcal{T}_0 \to G_4$ is the restriction of ϕ'.

2.4. Let R be a regular semisimple element in $\mathfrak{g}$. Let $\mathfrak{t}_R$ be the centralizer of R in $\mathfrak{g}$; let T_R be the centralizer of R in T. For any $z \in T_R$ we define a linear map $\Xi_{R,z} : \mathfrak{t}_R^\perp \to \mathfrak{g}/\mathfrak{t}_R^\perp$ by $\Xi_{R,z}(\xi) = [X, {}^z\xi] \mod \mathfrak{t}_R^\perp$ where X is any element of $\mathfrak{g}$ such that $\xi = [X, R]$. Note that such X exists; if X' is a another element such that $\xi = [X', R]$, then ${}^{z^{-1}}(X' - X) \in \mathfrak{t}_R$ hence $X' = X + \rho$ for some $\rho \in \mathfrak{t}_R$ and $[X', {}^z\xi] = [X, {}^z\xi] + [\rho, {}^z\xi]$. Since $[\rho, {}^z\xi] \in \mathfrak{t}_R^\perp$ we see that our map $\Xi_{R,z}$ is well defined.

2.5. Let $\mathcal{C} \subset \mathfrak{g}$ be the G-orbit of $-A_3$ for the adjoint action, a regular semisimple orbit. Let $\mathcal{Z}$ be the subset of G_4 consisting of all $y|R_1, R_2, R_3|$ such that

$R_3 \in \mathcal{C}$;

$y \in T_{R_3}$;

$R_2 + {}_xA_2 \in \mathfrak{t}_{R_3}^\perp$ where $Tx \in T\backslash G$ is uniquely determined by $R_3 = -{}_xA_3$;

$R_1 + {}_xA_1 + \Xi_{R_3,1}(R_2')/3 + \Xi_{R_3,y^{-1}}(R_2')/6 = 0$ in $\mathfrak{g}/\mathfrak{t}_{R_3}^\perp$ where $R_2' = R_2 + {}_xA_2$.

Note that $\mathcal{Z}$ is closed in G_4 (we use that $\mathcal{C}$ is closed in $\mathfrak{g}$). Moreover $\mathcal{Z}$ is a smooth subvariety of G_4. Indeed, $V = \{(y, R) \in G \times \mathcal{C}; y \in T_R\}$ is clearly smooth and $\mathcal{Z}$ is a fibration over V with fibres isomorphic to $\mathfrak{t}^\perp \times \mathfrak{t}^\perp$.

From the definitions we see that $\mathcal{T}_0 = \phi'^{-1}\mathcal{Z}$ and that the restriction of ϕ' defines a morphism $\underline{\phi}' : \mathcal{T}_0 \to \mathcal{Z}$ whose fibres are exactly the orbits of the free $\mathfrak{t}$-action on $\mathcal{T}_0$ given by

$$\tau : (Tx, y, X_1, R_1, R_2, R_3) \mapsto (Tx, y, X_1 + {}_x\tau, R_1, R_2, R_3).$$

Clearly, the local system $\tilde{\mathcal{E}}$ on $\mathcal{T}_0$ is the inverse image under $\underline{\phi}'$ of a local system on $\mathcal{Z}$ denoted again by $\tilde{\mathcal{E}}$. Next we show that

(a) *the function* $\tilde{h}_0 : \mathcal{T}_0 \to \mathbf{k}$ *is constant on each orbit of the* $\mathfrak{t}$*-action on* $\mathcal{T}_0$

that is, if $\tau \in \mathfrak{t}$ and $(Tx, y, X_1, R_1, R_2, R_3) \in \mathcal{T}_0$, then

$$\tilde{h}_0(Tx, y, X_1 + {}_x\tau, R_1, R_2, R_3) = \tilde{h}_0(Tx, y, X_1, R_1, R_2, R_3).$$

Thus, we must show that

$$\begin{aligned}
&\langle X_1 + {}_x\tau - {}_yX_1 - {}_x\tau, R_1\rangle + \langle {}_xA_2, [{}_yX_1 + {}_x\tau, X_1 + {}_x\tau]/2\rangle \\
&+ \langle {}_xA_3, [{}_yX_1 + {}_x\tau, [{}_yX_1 + {}_x\tau, X_1 + {}_x\tau]]/6 + [X_1 + {}_x\tau, [{}_yX_1 + {}_x\tau, X_1 + {}_x\tau]]/6\rangle \\
&= \langle X_1 - {}_yX_1, R_1\rangle + \langle {}_xA_2, [{}_yX_1, X_1]/2\rangle \\
&+ \langle {}_xA_3, [{}_yX_1, [{}_yX_1, X_1]]/6 + [X_1, [{}_yX_1, X_1]]/6\rangle.
\end{aligned}$$

(We have used that ${}_{xy}\tau = {}_x\tau$.) It is enough to show that

$$\begin{aligned}
&\langle {}_xA_2, [{}_x\tau, X_1]/2\rangle + \langle {}_xA_2, [{}_yX_1, {}_x\tau]/2\rangle \\
&+ \langle {}_xA_3, [{}_yX_1, [{}_yX_1, {}_x\tau]]/6 + [{}_yX_1, [{}_x\tau, X_1]]/6 + [{}_x\tau, [{}_yX_1, {}_x\tau]]/6 \\
&+ [{}_x\tau, [{}_yX_1, X_1]]/6 + [{}_x\tau, [{}_x\tau, X_1]]/6 + [X_1, [{}_yX_1, {}_x\tau]]/6 + [X_1, [{}_x\tau, X_1]]/6 \\
&+ [{}_x\tau, [{}_yX_1, X_1]]/6 + [{}_x\tau, [{}_yX_1, {}_x\tau]]/6 + [{}_x\tau, [{}_x\tau, X_1]]/6\rangle = 0.
\end{aligned}$$

Since $\langle {}_xA_i, [{}_x\tau, \xi]\rangle = 0$ for any $\xi \in \mathfrak{g}$, we see that it is enough to show

$$\langle {}_xA_3, [{}_yX_1, [{}_yX_1, {}_x\tau]]/6 + [{}_yX_1, [{}_x\tau, X_1]]/6 + [X_1, [{}_yX_1, {}_x\tau]]/6 \\ + [X_1, [{}_x\tau, X_1]]/6\rangle = 0.$$

It is enough to show the following two equalities:

(b) $$\langle {}_xA_3, [{}_yX_1, [{}_yX_1, {}_x\tau]] + [X_1, [{}_x\tau, X_1]]\rangle = 0,$$

(c) $$\langle {}_xA_3, [{}_yX_1, [{}_x\tau, X_1]] + [X_1, [{}_yX_1, {}_x\tau]]\rangle = 0.$$

The left hand side of (b) is

$$\langle {}_xA_3, {}_y[X_1, [X_1, {}_x\tau]] - [X_1, [X_1, {}_x\tau]]\rangle = \langle {}^{yx^{-1}}A_3 - {}^{x^{-1}}A_3, [X_1, [X_1, {}_x\tau]]\rangle$$

and this is zero since ${}^{yx^{-1}}A_3 = {}^{x^{-1}}A_3$. The left hand side of (c) is

$$\langle {}_xA_3, [{}_x\tau, [{}_yX_1, X_1]]$$

and this is zero since $\langle {}_xA_3, [{}_x\tau, \xi]\rangle = 0$ for any $\xi \in \mathfrak{g}$. This proves (a).

From (a) we see that there is a unique morphism $\hat{h} : \mathcal{Z} \to \mathbf{k}$ such that $\tilde{h}_0(s) = \hat{h}(\underline{\phi}'(s))$ for any $s \in \mathcal{T}_0$. It follows that $\mathcal{L}_{\tilde{h}_0} = \underline{\phi}'^*\mathcal{L}_{\hat{h}}$. Now $j : \mathcal{T}_0 \to G_4$ in 2.3 is a composition $\underline{j}\underline{\phi}'$ where $\underline{j} : \mathcal{Z} \to G_4$ is the imbedding. It follows that $j_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{\tilde{h}_0}) = \underline{j}_!(\tilde{\mathcal{E}} \otimes \underline{\phi}'_!\underline{\phi}'^*\mathcal{L}_{\hat{h}}) \cong \underline{j}_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{\hat{h}})[-2\delta]$. Combining with 2.3(a) we see that

$$\hat{K} \cong \underline{j}_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{\hat{h}})[-5\Delta - 4\delta].$$

Since $\mathcal{Z}$ is closed in G_4 and smooth, irreducible of dimension $3\Delta - 2\delta$ we see that $\underline{j}_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{\hat{h}})[3\Delta - 2\delta]$ is a simple perverse sheaf on G_4. Hence $\hat{K}[8\Delta + 2\delta]$ is a simple perverse sheaf on G_4 with support $\Sigma = \mathcal{Z}$. Using Laumon's theorem [La], it follows that

(b) $K[8\Delta + 2\delta]$ *is a simple perverse sheaf on* G_4.

3. The case $r = 3$

3.1. In this section we assume that $r = 3$. Now $\bar{\mathcal{X}}$ is the variety of all

$$(Tx, y, X_1, Y_1, R_1, R_2) \in (T\backslash G) \times G \times \mathfrak{g}^4$$

such that $xyx^{-1} \in T$, ${}_yX_1 - X_1 + Y_1 \in {}_x\mathfrak{b}$ and $R_2 = -{}_xA_2$. In our case we have

$$\hat{K} = \bar{\pi}_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{\bar{h}})[-2\Delta]$$

where $\bar{\pi} : \bar{\mathcal{X}} \to G_3$ and $\bar{h} : \bar{\mathcal{X}} \to \mathbf{k}$ are given by

$$\bar{\pi}(Tx, y, X_1, Y_1, R_1, R_2) = y|R_1, R_2|,$$

$$\bar{h}(Tx, y, X_1, Y_1, R_1, R_2) = \langle Y_1, R_1\rangle + \langle {}_xA_1, {}_yX_1 - X_1 + Y_1\rangle$$
$$+ \langle {}_xA_2, [{}_yX_1, Y_1]/2 - [{}_yX_1, X_1]/2 - [Y_1, X_1]/2\rangle.$$

Let

$$\bar{\mathcal{X}}' = \{(Tx, y, X_1, R_1, R_2, \beta); (Tx, y) \in (T\backslash G) \times G, (X_1, R_1, R_2) \in \mathfrak{g}^3, \beta \in {}_x\mathfrak{b},$$
$$xyx^{-1} \in T, R_2 = -{}_xA_2\}.$$

We define an isomorphism $\bar{\mathcal{X}} \xrightarrow{\sim} \bar{\mathcal{X}}'$ by

$$(Tx, y, X_1, Y_1, R_1, R_2) \mapsto (Tx, y, X_1, R_1, R_2, \beta)$$

where $\beta \in {}_x\mathfrak{b}$ is given by $\beta = {}_yX_1 - X_1 + Y_1$. We identify $\bar{\mathcal{X}} = \bar{\mathcal{X}}'$ via this isomorphism. Then $\bar{\pi}, \bar{h}$ become

$$\bar{\pi}(Tx, y, X_1, R_1, R_2, \beta) = y|R_1, R_2|,$$

$$\bar{h}(Tx, y, X_1, R_1, R_2, \beta) = \langle X_1 - {}_yX_1 + \beta, R_1\rangle + \langle {}_xA_1, \beta\rangle$$
$$+ \langle {}_xA_2, [{}_yX_1, X_1 - {}_yX_1 + \beta]/2 - [{}_yX_1, X_1]/2 - [X_1 - {}_yX_1 + \beta, X_1]/2\rangle$$
$$= \langle X_1 - {}_yX_1, R_1\rangle + \langle {}_xA_1 + R_1 + [{}_xA_2, X_1 + {}_yX_1]/2, \beta\rangle + \langle {}_xA_2, [{}_yX_1, X_1]/2\rangle.$$

Let

$$Z = \{(Tx, y, X_1, R_1, R_2) \in (T\backslash G) \times G \times \mathfrak{g}^3; xyx^{-1} \in T, R_2 = -{}_xA_2\},$$

$$Z_0 = \{(Tx, y, X_1, R_1, R_2) \in Z; {}_xA_1 + R_1 + [{}_xA_2, X_1 + {}_yX_1]/2 \in {}_x\mathfrak{n}\}$$

Define $\pi'_0 : Z_0 \to G_3$, $\tilde{h}_0 : Z_0 \to \mathbf{k}$ by

$$\pi'_0(Tx, y, X_1, R_1, R_2) = y|R_1, R_2|,$$

$$\tilde{h}_0(Tx, y, X_1, R_1, R_2) = \langle X_1 - {}_yX_1, R_1\rangle + \langle {}_xA_2, [{}_yX_1, X_1]/2\rangle.$$

The map $\bar{\mathcal{X}}' \to Z$ given by $(Tx, y, X_1, R_1, R_2, \beta) \mapsto (Tx, y, X_1, R_1, R_2)$ is a vector bundle with fibres isomorphic to $\mathfrak{b}$. Applying 1.3(b) to this vector bundle we see that

$$\hat{K} = \pi'_{0!}(\tilde{\mathcal{E}} \otimes \mathcal{L}_{\tilde{h}_0})[-3\Delta - \delta]$$

where the inverse image of $\mathcal{E}$ under $Z_0 \to T$, $(Tx, y, X_1, R_1, R_2) \mapsto xyx^{-1}$ is denoted again by $\tilde{\mathcal{E}}$. (We have used that $2\dim\mathfrak{b} = \Delta + \delta$.)

For any $R \in \mathcal{C}$ (see 1.5) let T_R be the centralizer of R in G and let $\mathfrak{t}_R$ be the centralizer of R in $\mathfrak{g}$. Let $\mathcal{R} \subset \mathrm{Hom}(\mathfrak{t}, \mathbf{k}^*)$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$; for any $\alpha \in \mathcal{R}$ let $\mathfrak{g}^\alpha$ be the corresponding (1-dimensional) root subspace and let $e^\alpha : T \to \mathbf{k}^*$ be the correspondings root homomorphism.

Let $\mathcal{R}^+ = \{\alpha \in \mathcal{R}; \mathfrak{g}^\alpha \subset \mathfrak{n}\}$, $\mathcal{R}^- = \mathcal{R} - \mathcal{R}^+$. For $R \in \mathcal{C}$ let $\mathfrak{g}_R^- = \oplus_{a\in\mathcal{R}^-}{}_x\mathfrak{g}^\alpha$, $\mathfrak{g}^+{-}_R = \oplus_{a\in\mathcal{R}^+}{}_x\mathfrak{g}^\alpha$ (where $R = -{}_xA_2$); we have a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_R^- \oplus \mathfrak{t}_R \oplus \mathfrak{g}_R^+$. Hence for any $X \in \mathfrak{g}$ we can write uniquely $X = X_R^- + X_R^0 + X_R^+$ with $X_R^- \in \mathfrak{g}_R^-$, $X_R^0 \in \mathfrak{t}_R$, $X_R^+ \in \mathfrak{g}_R^+$. Let $\hat{Z}$ be the variety of all (y, X, R_1, R) where $R \in \mathcal{C}$, $R_1 \in \mathfrak{g}$, $y \in T_R$, $X \in \mathfrak{g}_R^-$ such that ${}_xA_1 + R_1 + [{}_xA_2, X + {}_yX]/2 \in {}_x\mathfrak{n}$ for some/any $x \in G$ such that $R = -{}_xA_2$. Define $\hat{\pi} : \hat{Z} \to G_3$, $\hat{h} : \hat{Z} \to \mathbf{k}$ by $\hat{\pi}(y, X, R_1, R) = y|R_1, R|$,

$$\hat{h}(y, X, R_1, R) = \langle X - {}_yX, R_1\rangle.$$

We define $\zeta : Z_0 \to \hat{Z}$ by $(Tx, y, X, R_1, R) \mapsto (y, X_R^-, R_1, R)$. This is well defined since, if $\beta \in \mathfrak{b}$, $R = -{}_xA_2$ and $y \in T_R$, then $[{}_xA_2, {}_x\beta + {}_y({}_x\beta)] \in {}_x\mathfrak{n}$. Now ζ is a vector bundle with fibres isomorphic to $\mathfrak{b}$. Note also that $\pi'_0 = \hat{\pi}\zeta$. We show that $\tilde{h}_0 = \hat{h}\zeta$.

For a fixed $(Tx, y, X, R_1, R) \in Z_0$ we have ${}_xA_1 + R_1 + [{}_xA_2, X + {}_yX]/2 \in {}_x\mathfrak{n}$ and in particular

$$R_1^0 + {}_xA_1 = 0,$$

(a)
$$R_1^- = -[{}_xA_2, X^- + {}_y(X^-)]/2$$

where we write X^+, X^-, X^0 instead of X_R^+, X_R^-, X_R^0. We must show that

$$\langle X - {}_yX, R_1\rangle + \langle {}_xA_2, [{}_yX, X]/2\rangle = \langle X^- - {}_y(X^-), R_1\rangle$$

or equivalently

$$\langle X^+ + X^0 - {}_y(X^+ + X^0), R_1\rangle + \langle {}_xA_2, [{}_y(X^+ + X^0), X^-]/2 + [{}_y(X^-), X^+ + X^0]/2 + [{}_y(X^+ + X^0), X^+ + X^0]/2\rangle = 0,$$

that is,

$$\langle X^+ - {}_y(X^+), R_1^-\rangle + \langle {}_xA_2, [{}_y(X^+), X^-]/2 + [{}_y(X^-), X^+]/2\rangle = 0.$$

In the left hand side we replace R_1^- by the expression (a) and we obtain

$$\begin{aligned}&\langle X^+ - {}_y(X^+), -[{}_xA_2, X^- + {}_y(X^-)]/2\rangle + \langle {}_xA_2, [{}_y(X^+), X^-]/2 + [{}_y(X^-), X^+]/2\rangle \\ &= \langle {}_xA_2, [X^- + {}_y(X^-), X^+ - {}_y(X^+)]/2 + [{}_y(X^+), X^-]/2 + [{}_y(X^-), X^+]/2\rangle \\ &= \langle {}_xA_2, [X^-, X^+] - [{}_y(X^-), {}_y(X^+)]/2\rangle = \langle {}^{yx^{-1}}A_2 - {}^{x^{-1}}A_2, [X^-, X^+]\rangle = 0\end{aligned}$$

since ${}^{yx^{-1}}A_2 - {}^{x^{-1}}A_2 = 0$. Thus our claim is proved. Applying 1.3(b) to the vector bundle ζ we deduce

$$\hat{K} = K'[-4\Delta - 2\delta], \quad K' = \hat{\pi}_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{\hat{h}}),$$

where the inverse image of $\mathcal{E}$ under $\hat{Z} \to T$, $(y, X, R_1, R) \mapsto xyx^{-1}$ (where $R = -_x A_2$) is denoted again by $\tilde{\mathcal{E}}$. (We have used that $2\dim \mathfrak{b} = \Delta + \delta$.)

3.2. Let $\mathcal{Z}$ be the set of all $y|R_1, R| \in G_3$ such that $R \in \mathcal{C}$, $y \in T_R$ and $(R_1)^0_R = -_x A_1$ where $R = -_x A_2$. This is clearly a closed, smooth subvariety of G_3; it is irreducible of dimension $2\Delta - \delta$. For any $R \in \mathcal{C}$ let $\mathcal{Z}_R$ be the inverse image of R under the map $G_3 \to \mathcal{C}$, $y|R_1, R| \mapsto R$.

Let $\mathcal{H}^i_{y,R_1,R}$ be the stalk at $y|R_1, R| \in G_r$ of the i-th cohomology sheaf of K', see 3.1. We want to describe the vector spaces $\mathcal{H}^i_{y,R_1,R}$. Note that $\mathcal{H}^i_{y,R_1,R} = 0$ unless $y|R_1 R| \in \mathcal{Z}$; we now assume that this condition is satisfied. Using G-equivariance and the G-homogeneity of $\mathcal{C}$, we see that we may also assume that $R = -A_2$ and we write $\mathcal{H}^i_{y,R_1}$ instead of $\mathcal{H}^i_{y,R_1,R}$. We have

$$\mathcal{H}^i_{y,R_1} = H^i_c(\hat{\pi}^{-1}(y|R_1, R|), \tilde{\mathcal{E}} \otimes \mathcal{L}_{\hat{h}}).$$

For any $X \in \mathfrak{g}$ we can write uniquely $X = X^0 + \sum_{\alpha \in \mathcal{R}} X^\alpha$ where $X^0 \in \mathfrak{t}$, $X^\alpha \in \mathfrak{g}^\alpha$. Note that we have ${}_yX = X^0 + \sum_\alpha e^\alpha(y^{-1})X^\alpha$.

Then $\hat{\pi}^{-1}(y|R_1 R|)$ can be identified with the affine space

(a) $$\{(X^{-\alpha})_{\alpha \in \mathcal{R}^+}; \alpha(A_2)(1 + e^\alpha(y))X^{-\alpha}/2 = R_1^{-\alpha}\}.$$

The restriction of $\hat{h}$ to $\hat{\pi}^{-1}(y|R_1 R|)$ becomes the affine linear function

(b) $$(X^{-\alpha})_{\alpha \in \mathcal{R}^+} \mapsto \sum_{a \in \mathcal{R}^+} (1 - e^\alpha(y))\langle X^{-\alpha}, R_1^\alpha\rangle.$$

We consider several cases.

(1) for some $\alpha \in \mathcal{R}^+$ we have $1 + e^\alpha(y) = 0$ and $R_1^{-\alpha} \neq 0$;

(2) for any $\alpha \in \mathcal{R}^+$ such that $1 + e^\alpha(y) = 0$ we have $R_1^{-\alpha} = 0$ but for some such α we have $R_1^\alpha \neq 0$;

(3) for any $\alpha \in \mathcal{R}^+$ such that $1 + e^\alpha(y) = 0$ we have $R_1^{-\alpha} = 0$ and $R_1^\alpha = 0$;

In case (1), the affine space (a) is empty and $\mathcal{H}^i_{y,R_1} = 0$.

In case (2), the affine space (a) is nonempty and (b) is non-constant hence $\mathcal{H}^i_{y,R_1} = 0$.

For any $y \in T$ we set $\Xi_y = \{\alpha \in \mathcal{R}^+; 1 + e^\alpha(y) = 0\}$. In case (3), the affine space (a) is nonempty of dimension equal to $\sharp(\Xi_y)$ and (b) is constant, hence $\mathcal{H}^i_{y,R_1}$ is 1-dimensional if $i = 2\sharp(\Xi_y)$ and is 0 $i \neq 2\sharp(\Xi_y)$.

3.3. For any subset Ξ of $\mathcal{R}^+$ let $T^\Xi = \{y \in T; \Xi_y = \Xi\}$ (the sets T^Ξ form a partition of T). Note that $T^\emptyset$ is an open dense subset of T. For $\Xi \subset \mathcal{R}^+$ let $\mathcal{Z}_R^\Xi$ be the set of all $y|R_1 R| \in \mathcal{Z}_R$ such that $y \in T^\Xi$ and $R_1^\alpha = 0$, $R_1^{-\alpha} = 0$ for all $\alpha \in \Xi$. The subsets $\mathcal{Z}_R^\Xi$ are clearly disjoint. Let $\mathcal{Z}'_R = \mathcal{Z}_R - \cup_{\Xi \subset \mathcal{R}^+} \mathcal{Z}_R^\Xi$. Note that for $y|R_1, R| \in \mathcal{Z}_R^\Xi$, $\mathcal{H}^i_{y,R_1}$ is 1-dimensional if $i = 2\sharp(\Xi)$ and is 0 if $i \neq 2\sharp(\Xi)$. Moreover, for $y|R_1, R| \in \mathcal{Z}'_R$, we have $\mathcal{H}^i_{y,R_1} = 0$ for all i. We show that for any $\Xi \subset \mathcal{R}^+$ we have

(a) $\dim \mathcal{X}_R^\Xi + 2\sharp(\Xi) \le \dim \mathcal{Z}_R$ *with strict inequality unless* $\Xi = \emptyset$.

Indeed, we have $\dim \mathcal{X}_R^\Xi = \dim T^\Xi + 2\sharp(\mathcal{R}^+ - \Xi)$. On the other hand, $\dim \mathcal{Z}_R = \delta + 2\sharp(\mathcal{R}^+)$. Thus (a) is equivalent to $\dim T^\Xi \le \delta$, with strict inequality unless $\Xi = \emptyset$; this is obvious.

From (a) we see that $K'|_{\mathcal{Z}_R}$ satisfies half of the defining properties of an intersection cohomology complex (the ones not involving Verdier duality). It follows that $K'|_{\mathcal{Z}}$ itself satisfies the same half of the defining properties of an intersection cohomology complex; moreover Σ (the support of K') is equal to $\mathcal{Z}$. Hence the perverse cohomology sheaves of $K'[2\Delta - \delta]$ satisfy ${}^pH^i(K'[2\Delta - d]) = 0$ for $i > 0$ and ${}^pH^0(K'[2\Delta - d])$ is a simple perverse sheaf on G_3. Since $\hat{K} = K'[-4\Delta - 2\delta]$, it follows that ${}^pH^i(\hat{K}[6\Delta + \delta]) = 0$ for $i > 0$ and ${}^pH^0(\hat{K}[6\Delta + \delta])$ is a simple perverse sheaf on G_3. Using Laumon's theorem [La] we deduce:

(b) ${}^pH^i(K[6\Delta + \delta]) = 0$ *for* $i > 0$ *and* ${}^pH^0(K[6\Delta + \delta])$ *is a simple perverse sheaf on* G_3.

3.4. Let $\mathcal{Z}^\emptyset$ be the set of all $y|R_1, R| \in \mathcal{Z}$ such that for any $\alpha \in \mathcal{R}^+$ we have $e^\alpha(xyx^{-1}) \neq -1$ (where $R = -{}_xA_2$); this is an open dense subset of $\mathcal{Z}$. We define $f : \mathcal{Z}^\emptyset \to \mathbf{k}$ by

$$f(y|R_1, R|) = \sum_{\alpha \in \mathcal{R}^+} \frac{2}{\alpha(A_2)} \frac{1 - e^\alpha(xyx^{-1})}{1 + e^\alpha(xyx^{-1})} \langle ({}_xR_1)^\alpha, ({}_xR_1)^{-\alpha} \rangle$$

$$= \sum_{\alpha \in \mathcal{R}} \frac{2}{\alpha(A_2)} \frac{1}{1 + e^\alpha(xyx^{-1})} \langle ({}_xR_1)^\alpha, ({}_xR_1)^{-\alpha} \rangle. \tag{a}$$

For $(y, X, R_1, R) \in \hat{\pi}^{-1}(\mathcal{Z}^\emptyset)$ we have

$$f(\hat{\pi}(y, X, R_1, R)) = \hat{h}(y, X, R_1, R). \tag{b}$$

To prove (b) we can assume that $R = -A_2$. We then have

$$\hat{h}(y, X, R_1, R) = \sum_{a \in \mathcal{R}^+} (1 - e^\alpha(y)) \langle X^{-\alpha}, R_1^\alpha \rangle.$$

Replacing here $X^{-\alpha}$ by $\frac{2}{\alpha(A_2)(1+e^\alpha(y))} R_1^{-\alpha}$ we obtain

$$\hat{h}(y, X, R_1, R) = \sum_{a \in \mathcal{R}^+} (1 - e^\alpha(y)) \frac{2}{\alpha(A_2)(1 + e^\alpha(y))} \langle R_1^{-\alpha}, R_1^\alpha \rangle = f(y|R_1, R|).$$

as required. Since $\hat{h}$ is an isomorphism $\hat{\pi}^{-1}(\mathcal{Z}^\emptyset) \xrightarrow{\sim} \mathcal{Z}^\emptyset$ (by results in 3.3), we see that $K'|_{\mathcal{Z}^\emptyset}$ is the rank 1 local system $\tilde{\mathcal{E}} \otimes \mathcal{L}_f$ on $\mathcal{Z}^\emptyset$ where the inverse image of $\mathcal{E}$ under $\mathcal{Z}^\emptyset \to T$, $y|R_1, R| \mapsto xyx^{-1}$ (where $R = -{}_xA_2$) is denoted again by $\tilde{\mathcal{E}}$. It follows that the simple perverse sheaf ${}^pH^0(\hat{K}[6\Delta+\delta])$ on G_3 is associated to the local system $\tilde{\mathcal{E}} \otimes \mathcal{L}_f$ on the locally closed smooth irreducible subvariety $\mathcal{Z}^\emptyset$ of G_3.

3.5. It is likely that $K'[2\Delta - d]$ is a simple perverse sheaf on G_3. This would imply that $K[6\Delta+\delta]$ is a simple perverse sheaf on G_3.

4. A comparison of two complexes

4.1. We preserve the assumptions in 1.4. Let L be as in 0.2 where $f_i : \mathfrak{t} \to \mathbf{k}$ is $\tau \mapsto \langle A_i, \tau\rangle$ for $i = 1, \dots, r-1$. In this section we describe a strategy for showing that a shift of L is isomorphic to K in 1.4.

We define a sequence of algebraic varieties $\mathcal{X}_r, \mathcal{X}_{r-1}, \dots, \mathcal{X}_{r-2r'}$ as follows. For $i \in \{r-r', r-r'+1, \dots, r\}$ let $\mathcal{X}_i$ be the variety consisting of all

$(Tx, y, X_1, \dots, X_{r-1}, Y_1, \dots, Y_{r-1}) \in (T\backslash G) \times G \times \mathfrak{g}^{2r-2}$

such that $xyx^{-1} \in B$ and

$u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j) \in {}_x\mathfrak{b}$ for $1 \le j \le i-1$.

For $i \in \{r-2r', r-2r'+1, \dots, r-r'-1\}$ let $\mathcal{X}_i$ be the variety of all

$(Tx, y, X_1, \dots, X_{r-1}, Y_1, \dots, Y_{r-1}) \in (T\backslash G) \times G \times \mathfrak{g}^{2r-2}$

such that $xyx^{-1} \in T$ and

$u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j) \in {}_x\mathfrak{t}$ for $1 \le j \le r-r'-i-1$,

$u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j) \in {}_x\mathfrak{b}$ for $r-r'-i \le j \le r-r'-1$.

For $i = r, r-1, \dots, r-2r'$ we have a diagram

$$G_r \xleftarrow{\pi_i} \mathcal{X}_i \xrightarrow{h_i} \mathbf{k}$$

where $\pi_i(Tx, y, X_1, \dots, X_{r-1}, Y_1, \dots, Y_{r-1}) = y|Y_1, Y_2, \dots, Y_{r-1}|$,

$$h_i(Tx, y, X_1, \dots, X_{r-1}, Y_1, \dots, Y_{r-1})$$
$$= \sum_{j\in[1,r-1]} \langle {}_xA_j, u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j)\rangle.$$

We define $\iota : \mathcal{X}_i \to T$ by $\iota(Tx, y, X_1, \dots, X_{r-1}, Y_1, \dots, Y_{r-1}) = d(xyx^{-1})$ where $d : B \to T$ is as in 0.1. Let $\tilde{\mathcal{E}} = \iota^*\mathcal{E}$. The inverse image of $\tilde{\mathcal{E}}$ under various maps to $\mathcal{X}_i$ is denoted again by $\tilde{\mathcal{E}}$.

Let $L_i = \pi_{i!}(\tilde{\mathcal{E}} \otimes \mathcal{L}_{h_i}) \in \mathcal{D}(G_r)$. Let H be the kernel of the obvious map $B_r \to T$, a connected unipotent group of dimension $r(\Delta+\delta)/2 - \delta$. Note that $\mathcal{X}_r$ is a principal H-bundle over $\tilde{G}_r$ in 0.2. It follows that $L_r \cong K[r(\Delta+\delta)-2\delta]$. On the other hand we have $L = L_{r-2r'}$. We would like to show that $L \cong K[r(\Delta+\delta)-2\delta]$. It is enough to show that

(a) $L_r = L_{r-1} = \dots = L_{r-2r'}$.

Note that $\mathcal{X}_r \subset \mathcal{X}_{r-r'} \subset \dots \subset \mathcal{X}_{r'} \supset \mathcal{X}_{r'-1} \supset \dots \supset \mathcal{X}_{r-2r'}$. For $r' \le i \le r-1$ let

$\pi'_i : \mathcal{X}_i - \mathcal{X}_{i+1} \to G_r$, $h'_i : \mathcal{X}_i - \mathcal{X}_{i+1} \to \mathbf{k}$ be the restrictions of π_i, h_i to $\mathcal{X}_i - \mathcal{X}_{i+1}$; let $L'_i = p'_{i!}(\tilde{\mathcal{E}} \otimes \mathcal{L}_{h'_i}) \in \mathcal{D}(G_r)$.

For $r - 2r' + 1 \le i \le r'$ let $\pi''_i : \mathcal{X}_i - \mathcal{X}_{i-1} \to G_r$, $h''_i : \mathcal{X}_i - \mathcal{X}_{i-1} \to \mathbf{k}$ be the restrictions of π_i, h_i to $\mathcal{X}_i - \mathcal{X}_{i-1}$; let $L''_i = p''_{i!}(\tilde{\mathcal{E}} \otimes \mathcal{L}_{h''_i}) \in \mathcal{D}(G_r)$.

From the definitions we have distinguished triangles (L'_i, L_i, L_{i+1}) (for $i = r', r'+1, \dots, r-1$) and (L''_i, L_i, L_{i-1}) (for $r - 2r' + 1 \le i \le r'$). Hence (a) would follow from statements (b),(c) below.

(b) $L''_{r'} = L''_{r'-1} = \dots = L''_{r-2r'+1} = 0$.

(c) $L'_{r'} = L'_{r'+1} = \dots = L'_{r-1} = 0$.

Here is a strategy to prove (b),(c).

For $r - 2r' + 1 \le i \le r'$ one should partition $\mathcal{X}_i - \mathcal{X}_{i-1}$ into pieces isomorphic to $\mathfrak{g}$ so that the restriction of h''_i to each piece is a nonconstant affine linear function and the restriction of $\tilde{\mathcal{E}}$ is $\bar{\mathbf{Q}}_l$. For $r' \le i \le r-1$ one should partition $\mathcal{X}_i - \mathcal{X}_{i+1}$ into pieces isomorphic to an affine space so that the restriction of h'_i to each piece is a nonconstant affine linear function and the restriction of $\tilde{\mathcal{E}}$ is $\bar{\mathbf{Q}}_l$. This should give the desired result. In 4.2-4.5 we carry out this strategy in several cases which are sufficient to deal with the cases where $r \in \{2, 3, 4\}$.

4.2. In this subsection we show that

(a) $L''_{r'} = 0$.

Note that $\mathcal{X}_{r'} - \mathcal{X}_{r'-1}$ is the set of all

$(Tx, y, X_1, \dots, X_{r-1}, Y_1, \dots, Y_{r-1}) \in (T\backslash G) \times G \times \mathfrak{g}^{2r-2}$

such that $xyx^{-1} \in B - T$ and

$u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j) \in {}_x\mathfrak{b}$ for $1 \le j \le r - r' - 1$.

Let Z be the set of all

$(Tx, y, X_1, \dots, X_{r-2}, Y_1, \dots, Y_{r-1}) \in (T\backslash G) \times G \times \mathfrak{g}^{2r-3}$

such that $xyx^{-1} \in B - T$ and

$u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j) \in {}_x\mathfrak{b}$ for $1 \le j \le r - r' - 1$.

Now $\pi''_{r'}$ is a composition $\mathcal{X}_{r'} - \mathcal{X}_{r'-1} \xrightarrow{a} Z \xrightarrow{a'} G_r$ where

$a(Tx, y, X_1, \dots, X_{r-1}, Y_1, \dots, Y_{r-1}) = (Tx, y, X_1, \dots, X_{r-2}, Y_1, \dots, Y_{r-1})$,

$a'(Tx, y, X_1, \dots, X_{r-2}, Y_1, \dots, Y_{r-1}) = y|Y_1, \dots, Y_{r-1}|$.

It is enough to show that $a_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{h''_{r'}}) = 0$. Clearly, $\tilde{\mathcal{E}}$ is the inverse image under a of a local system on Z denoted again by $\tilde{\mathcal{E}}$. Hence

$$a_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{h''_{r'}}) = \tilde{\mathcal{E}} \otimes a_!(\mathcal{L}_{h''_{r'}})$$

and it is enough to show that $a_!(\mathcal{L}_{h''_{r'}}) = 0$. It is also enough to show that for any $s = (Tx, y, X_1, \dots, X_{r-2}, Y_1, \dots, Y_{r-1}) \in Z$ we have $H^*_c(a^{-1}(s), \mathcal{L}_{h''_{r'}}) = 0$. Now $a^{-1}(s)$ may be identified with the affine space $\mathfrak{g}$ with coordinate X_{r-1} and $h''_{r'}$ is of the form $X_{r-1} \mapsto \langle {}_xA_{r-1}, {}_yX_{r-1} - X_{r-1} \rangle + c$ where c is a constant (for fixed s). It is enough to show that the linear form

$$X_{r-1} \mapsto \langle {}_xA_{r-1}, {}_yX_{r-1} - X_{r-1} \rangle = \langle {}^{yx^{-1}}A_{r-1} - {}^{x^{-1}}A_{r-1}, X_{r-1} \rangle$$

on $\mathfrak{g}$ is not identically zero. If it was identically zero, we would have ${}^{yx^{-1}}A_{r-1} = {}^{x^{-1}}A_{r-1}$ hence xyx^{-1} centralizes A_{r-1} hence $xyx^{-1} \in T$ contradicting $xyx^{-1} \in B - T$. This proves (a).

4.3. In this subsection we show that

(a) $L''_{r'-1} = 0$ (assuming that $r = 4$).

Note that $\mathcal{X}_1 - \mathcal{X}_0$ is the set of all $(Tx, y, X_1, X_2, X_3, Y_1, Y_2, Y_3) \in (T\backslash G) \times G \times \mathfrak{g}^6$ such that $xyx^{-1} \in T$ and ${}_yX_1 - X_1 + Y_1 \in {}_x\mathfrak{b} - {}_x\mathfrak{t}$.

We have a free action of $\mathfrak{g}$ on $\mathcal{X}_1 - \mathcal{X}_0$:

$$E : (Tx, y, X_1, X_2, X_3, Y_1, Y_2, Y_3) \mapsto (Tx, y, X_1, X_2 + E, X_3 + [E, X_1], Y_1, Y_2, Y_3).$$

The orbit of $(Tx, y, X_1, 0, X_3, Y_1, Y_2, Y_3)$ is

$$\mathcal{O} = \{(Tx, y, X_1, E, X_3 + [E, X_1], Y_1, Y_2, Y_3), E \in \mathfrak{g}\}.$$

It is enough to show that $H_c^*(\mathcal{O}, \tilde{\mathcal{E}} \otimes \mathcal{L}_{h_1''}) = 0$ for any such $\mathcal{O}$. Clearly $\tilde{\mathcal{E}} \cong \bar{\mathbf{Q}}_l$ on $\mathcal{O}$. Hence it is enough to show that $H_c^*(\mathcal{O}, \mathcal{L}_{h_1''}) = 0$. We can identify $\mathcal{O}$ with the affine space $\mathfrak{g}$ with coordinate E. On this affine space h_1'' is of the form

$$E \mapsto \langle {}_xA_2, {}_yE - E\rangle + \langle {}_xA_3, {}_y[E, X_1] - [E, X_1] + [{}_yE, Y_1] + [E, X_1] - [{}_yE, X_1]\rangle + c$$

where c is a constant (for our fixed $\mathcal{O}$). We have

$$\langle {}_xA_2, {}_yE - E\rangle = \langle {}^{yx^{-1}}A_2 - {}^{x^{-1}}A_2, E\rangle = 0$$

since ${}^{yx^{-1}}A_2 = {}^{x^{-1}}A_2$ (recall that $xyx^{-1} \in T$). Hence h_1'' is of the form

$$E \mapsto \langle {}_xA_3, [{}_yE, {}_yX_1] + [{}_yE, Y_1] - [{}_yE, X_1]\rangle + c = \langle {}_yE, [\xi, {}_xA_3]\rangle + c$$

where $\xi = {}_yX - X_1 + Y_1$. It is enough to show that the linear form $E \mapsto \langle {}_yE, [\xi, {}_xA_3]\rangle$ on $\mathfrak{g}$ is not identically zero. If it is identically zero we would have $[\xi, {}_xA_3] = 0$ that is, ξ is in the centralizer of ${}_xA_3$ so that $\xi \in {}_x\mathfrak{t}$, contradicting $\xi \in {}_x\mathfrak{b} - {}_x\mathfrak{t}$. This proves (a).

4.4. In this subsection we show that

(a) $L'_{r-1} = 0$.

Note that $\mathcal{X}_{r-1} - \mathcal{X}_r$ is the set of all

$(Tx, y, X_1, X_2, \dots, X_{r-1}, Y_1, Y_2, \dots, Y_{r-1}) \in (T\backslash G) \times G \times \mathfrak{g}^{2r-2}$

such that $xyx^{-1} \in B$ and

$u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j) \in {}_x\mathfrak{b}$ for $j = 1, 2, \dots, r-2$,

$u_j({}_yX_1, \dots, {}_yX_j, Y_1, \dots, Y_j, X_1, \dots, X_j) \notin {}_x\mathfrak{b}$ for $j = r-1$.

Now π'_{r-1} is a composition $\mathcal{X}_{r-1} - \mathcal{X}_r \xrightarrow{a} Z \xrightarrow{a'} G_r$ where Z is the set of all

$$(Bx, y, X_1, X_2, \dots, X_{r-1}, Y_1, Y_2, \dots, Y_{r-1}) \in (B\backslash G) \times G \times \mathfrak{g}^{2r-2}$$

satisfying the same conditions as the points of $\mathcal{X}_{r-1}-\mathcal{X}_r$ and a is the obvious map. It is enough to show that $a_!(\tilde{\mathcal{E}}\otimes\mathcal{L}_{h'_{r-1}})=0$. Clearly $\tilde{\mathcal{E}}$ is the inverse image under a of a local system on Z denoted again by $\tilde{\mathcal{E}}$. Hence $a_!(\tilde{\mathcal{E}}\otimes\mathcal{L}_{h'_{r-1}})=\tilde{\mathcal{E}}\otimes a_!(\mathcal{L}_{h'_{r-1}})$ and it is enough to show that $a_!(\mathcal{L}_{h'_{r-1}})=0$. It is also enough to show that for any $s=(Bx,y,X_1,X_2,\dots,X_{r-2},Y_1,Y_2,\dots,Y_{r-1})\in Z$ we have $H_c^*(a^{-1}(s),\mathcal{L}_{h'_{r-1}})=0$. Now $a^{-1}(s)$ may be identified with U by

$$u\mapsto(Tux,y,X_1,X_2,\dots,X_{r-1},Y_1,Y_2,\dots,Y_{r-1})$$

where x is a fixed representative of Bx. For $j\in[1,r-1]$ we set

$$\xi_j={}^xu_j({}_yX_1,\dots,{}_yX_j,Y_1,\dots,Y_j,X_1,\dots,X_j)\in\mathfrak{g}.$$

Then h'_{r-1} becomes the function $U\to\mathbf{k}$ given by

$$\begin{aligned}&u\mapsto\sum_{j\in[1,r-1]}\langle{}_{ux}A_j,{}_x\xi_j\rangle\\&=\sum_{j\in[1,r-2]}(\langle A_j,{}^u\xi_j-\xi_j\rangle+\langle A_j,\xi_j\rangle)+\langle A_{r-1},\xi_{r-1}\rangle+\langle{}_uA_{r-1}-A_{r-1},\xi_{r-1}\rangle.\end{aligned}$$

For $j\in[1,r-2]$ we have $\xi_j\in\mathfrak{b}$ hence ${}^u\xi_j-\xi_j\in\mathfrak{n}$ so that $\langle A_j,{}^u\xi_j-\xi_j\rangle=0$. Thus h'_{r-1} becomes the function $U\to\mathbf{k}$ given by

$$u\mapsto\langle{}_uA_{r-1}-A_{r-1},\xi_{r-1}\rangle+c$$

where c is a constant (for fixed s). We identify U with $\mathfrak{n}$ by $u\mapsto{}_uA_{r-1}-A_{r-1}$. Then h'_{r-1} becomes the function $\mathfrak{n}\to\mathbf{k}$ given by $\zeta\mapsto\langle\zeta,\xi_{r-1}\rangle+c$. This function is affine linear and nonconstant since $\xi_{r-1}\notin\mathfrak{b}=\mathfrak{n}^\perp$. It follows that $H_c^*(a^{-1}(s),\mathcal{L}_{h'_{r-1}})=0$ and (a) is proved.

4.5. In this subsection we show that
(a) $L'_{2r'-2}=0$ (assuming that $r=4$).
Note that $\mathcal{X}_2-\mathcal{X}_3$ is the set of all
$(Tx,y,X_1,X_2,X_3,Y_1,Y_2,Y_3)\in(T\backslash G)\times G\times\mathfrak{g}^6$
such that $xyx^{-1}\in B$,
(b) ${}_yX_1-X_1+Y_1\in{}_x\mathfrak{b}$,
(c) ${}_yX_2-X_2+Y_2+[{}_yX_1,Y_1]/2-[{}_yX_1,X_1]/2-[Y_1,X_1]/2\notin{}_x\mathfrak{b}$.
Let $Z=\{(Tx,y,Y_1,Y_2,Y_3)\in(T\backslash G)\times G\times\mathfrak{g}^3;xyx^{-1}\in B\}$. The inverse image of $\mathcal{E}$ under $Z\to T$, $(Tx,y,Y_1,Y_2,Y_3)\mapsto d(xyx^{-1})$ is denoted by $\tilde{\mathcal{E}}_0$.

Now π'_2 is a composition $\mathcal{X}_2-\mathcal{X}_3\xrightarrow{a}Z\xrightarrow{a'}G_r$ where

$$\begin{aligned}&a(Tx,y,X_1,X_2,X_3,Y_1,Y_2,Y_3)=(Tx,y,Y_1,Y_2,Y_3),\\&a'(Tx,y,Y_1,Y_2,Y_3)=y|Y_1,Y_2,Y_3|\end{aligned}$$

. We have $a^*\tilde{\mathcal{E}}_0 = \tilde{\mathcal{E}}$. It is enough to show that $a_!(\tilde{\mathcal{E}} \otimes \mathcal{L}_{h'_2}) = 0$ that is, $\tilde{\mathcal{E}}_0 \otimes a_!(\mathcal{L}_{h'_2}) = 0$. Thus it is enough to prove that $a_!(\mathcal{L}_{h'_2}) = 0$. Hence it is enough to show that for any $s = (Tx, y, Y_1, Y_2, Y_3) \in (T\backslash G) \times G \times \mathfrak{g}^3$ we have $H_c^*(a^{-1}(s), \mathcal{L}_{h'_2}) = 0$.

Let $\mathcal{G} = \{|E, E', E''| \in G_4; E \in \mathfrak{n}, E' \in \mathfrak{n}, E'' \in \mathfrak{n}\}$; this is a closed subgroup of G_4.

We fix a representative x in Tx and we define a free $\mathcal{G}$-action on $a^{-1}(s)$ by

$$\begin{aligned}&|E, E', E''| : (Tx, y, X_1, X_2, X_3, Y_1, Y_2, Y_3) \mapsto (Tx, y, X_1 + {}_xE,\\&X_2 + {}_xE' + [{}_xE, X_1]/2, X_3 + {}_xE''\\&+ [{}_xE', X_1] - [{}_xE, [{}_xE, X_1]]/6 - [X_1, [{}_xE, X_1]]/3, Y_1, Y_2, Y_3).\end{aligned}$$

We verify that this action is well defined (that is, the equations (b),(c) are preserved). To show that (b) is preserved it is enough to verify that ${}_{xy}E - {}_xE \in {}_x\mathfrak{b}$ or that ${}^{xy^{-1}x^{-1}}E - E \in \mathfrak{b}$; this follows from $E \in \mathfrak{n}$, $xyx^{-1} \in B$. To show that (c) is preserved it is enough to verify that

$$\begin{aligned}&{}_{xy}E' + [{}_{xy}E, {}_yX_1]/2 - {}_xE' - [{}_xE, X_1]/2 + [{}_{xy}E, Y_1]/2\\&- [{}_yX_1, {}_xE]/2 - [{}_{xy}E, X_1]/2 - [{}_{xy}E, {}_xE]/2 - [Y_1, {}_xE]/2 \in {}_x\mathfrak{b}\end{aligned}$$

(when (b) holds) or that

$$[{}_{xy}E, {}_yX_1 - X_1 + Y_1]/2 + [{}_xE, {}_yX_1 - X_1 + Y_1]/2 - [{}_{xy}E, {}_xE]/2 + {}_{xy}E' - {}_xE' \in {}_x\mathfrak{b}$$

and this follows from (b) and from ${}_{xy}E \in {}_x\mathfrak{b}$, ${}_xE \in {}_x\mathfrak{b}$, ${}_{xy}E' \in {}_x\mathfrak{b}$, ${}_xE' \in {}_x\mathfrak{b}$.

It is enough to show that for any $\mathcal{G}$-orbit $\mathcal{O}$ in $a^{-1}(s)$ we have $H_c^*(\mathcal{O}, \mathcal{L}_{h'_2}) = 0$. We may identify $\mathcal{O} = \mathcal{G}$ using a base point $(Tx, y, X_1, X_2, X_3, Y_1, Y_2, Y_3) \in \mathcal{O}$ (with a fixed representative x for Tx) and we identify $\mathcal{G} = \mathfrak{n}^3$ using $[E, E', E''] \leftrightarrow (E, E', E'')$. Then h'_2 becomes a function $h'' : \mathfrak{n}^3 \to \mathbf{k}$ of the following form (we have substituted $Y_1 = X_1 - {}_yX_1 + {}_x\beta$ where $\beta \in \mathfrak{b}$):

$$(E, E', E'') \mapsto h''(E, E', E'') = \langle {}_xA_1, \xi_1\rangle + \langle {}_xA_2, \xi_2 + \xi'_2\rangle + \langle {}_xA_3, \xi_3 + \xi'_3 + \xi''_3\rangle$$

where

$$\xi_1 = {}_x\beta + {}_{xy}E - {}_xE,$$

$$\begin{aligned}\xi_2 = {}&{}_yX_2 + [{}_{xy}E, {}_yX_1]/2 - X_2 - [{}_xE, X_1]/2 + Y_2\\&+ [{}_yX_1 + {}_{xy}E, X_1 - {}_yX_1 + {}_x\beta]/2 - [{}_yX_1 + {}_{xy}E, X_1 + {}_xE]/2\\&- [X_1 - {}_yX_1 + {}_x\beta, X_1 + {}_xE]/2,\end{aligned}$$

$$\xi'_2 = {}_{xy}E' - {}_xE'$$

$$\begin{aligned}
\xi_3 &= {}_yX_3 - X_3 + Y_3 - [{}_{xy}E, [{}_{xy}E, {}_yX_1]]/6 - [{}_yX_1, [{}_{xy}E, {}_yX_1]]/3\\
&+ [{}_xE, [{}_xE, X_1]]/6 + [X_1, [{}_xE, X_1]]/3\\
&+ [{}_yX_2 + [{}_{xy}E, {}_yX_1]/2, X_1 - {}_yX_1 + {}_x\beta] + [X_2 + [{}_xE, X_1]/2, X_1 + {}_xE]\\
&- [{}_yX_2 + [{}_{xy}E, {}_yX_1]/2, X_1 + {}_xE] - [Y_2, X_1 + {}_xE]\\
&- [{}_yX_1 + {}_{xy}E, [{}_yX_1 + {}_{xy}E, X_1 - {}_yX_1 + {}_x\beta]]/6\\
&- [X_1 - {}_yX_1 + {}_x\beta, [{}_yX_1 + {}_{xy}E, X_1 - {}_yX_1 + {}_x\beta]]/3\\
&+ [X_1 + {}_xE, [{}_yX_1 + {}_{xy}E, X_1 - {}_yX_1 + {}_x\beta]]/2\\
&+ [{}_yX_1 + {}_{xy}E, [{}_yX_1 + {}_{xy}E, X_1 + {}_xE]]/6\\
&+ [{}_yX_1 + {}_{xy}E, [X_1 - {}_yX_1 + {}_x\beta, X_1 + {}_xE]]/6\\
&+ [X_1 - {}_yX_1 + {}_x\beta, [{}_yX_1 + {}_{xy}E, X_1 + {}_xE]]/6\\
&+ [X_1 - {}_yX_1 + {}_x\beta, [X_1 - {}_yX_1 + {}_x\beta, X_1 + {}_xE]]/6\\
&- [X_1 + {}_xE, [{}_yX_1 + {}_{xy}E, X_1 + {}_xE]]/3\\
&- [X_1 + {}_xE, [X_1 - {}_yX_1 + {}_x\beta, X_1 + {}_xE]]/3\rangle,
\end{aligned}$$

$$\begin{aligned}
\xi_3' &= {}_{xy}E'' + [{}_{xy}E', {}_yX_1] - {}_xE'' - [{}_xE', X_1]\\
&+ [{}_{yx}E', X_1 - {}_yX_1 + {}_x\beta] + [{}_xE', X_1] - [{}_{xy}E', X_1],
\end{aligned}$$

$$\xi_3'' = [{}_xE', {}_xE] - [{}_{xy}E', {}_xE].$$

It is enough to show that for any fixed E', E'' in $\mathfrak{n}$, the function $E \mapsto h_1''(E) = h''(E, E', E'')$ is affine linear and nonconstant. Let

$$S = {}_yX_2 - X_2 + Y_2 + [{}_yX_1, Y_1]/2 - [{}_yX_1, X_1]/2 - [Y_1, X_1]/2.$$

A computation shows that

$$\xi_1 - C_1 \in {}_x\mathfrak{n}, \xi_2 - C_2 \in {}_x\mathfrak{n}, \xi_3 - [{}_xE, S] - C_3 \in {}_x\mathfrak{n}, \xi_3' = C_4$$

where C_1, C_2, C_3, C_4 are vectors in $\mathfrak{g}$ independent of E. Moreover, $\xi_2' \in {}_x\mathfrak{n}$, $\xi_3'' \in {}_x\mathfrak{n}$. Since $\langle {}_xA_i, {}_x\mathfrak{n}\rangle = 0$, for some constant $c \in \mathbf{k}$ we have

$$h_1''(E) = \langle {}_xA_3, [{}_xE, S]\rangle + c = \langle S, [{}_xA_3, {}_xE]\rangle + c.$$

In particular, $E \mapsto h_1''(E)$ is affine linear on $\mathfrak{n}$. To show that it is nonconstant it is enough to show that $E \mapsto \langle S, [{}_xA_3, {}_xE]\rangle$ is not identically zero. Assume that it is identically zero. Since $E \mapsto [A_3, E]$ is a vector space isomorphism $\mathfrak{n} \xrightarrow{\sim} \mathfrak{n}$ it would follow that $\langle S, {}_x\tilde{E}\rangle = 0$ for any $\tilde{E} \in \mathfrak{n}$ hence $S \in {}_x(\mathfrak{n}^\perp)$ that is, $S \in {}_x\mathfrak{b}$. This contradicts the definition of $\mathcal{X}_2 - \mathcal{X}_3$ and proves (a).

4.6. In this subsection we assume that $r \in \{2, 3, 4\}$. From 4.2, 4.3, 4.4, 4.5 we see that 4.1(b),(c) hold. Hence 4.1(a) holds. Hence $L \cong K[2\Delta]$ if $r = 2$, $L \cong K[3\Delta+\delta]$ if $r = 3$ and $L \cong K[4\Delta + 2\delta]$ if $r = 4$.

Using now 2.1(b), 2.5(b), 3.3(b) we deduce the folowing result.

Theorem 4.7. *(a) $L[r\Delta]$ is a simple perverse sheaf on G_r provided that $r = 2$ or $r = 4$.*

(b) If $r = 3$ we have ${}^pH^i(L[r\Delta]) = 0$ for $i > 0$ and ${}^pH^0(L[r\Delta]) = 0$ is a simple perverse sheaf on G_r.

It is likely that in fact $L[r\Delta]$ is a simple perverse sheaf on G_r for any $r \geq 2$. For $r = 3$ this would follow if the truth of the statements in 3.5 could be established.

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